On a Class of Singular Douglas and Projectively flat Finsler Metrics

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Abstract

Singular Finsler metrics, such as Kropina metrics and m-Kropina metrics, have a lot of applications in the real world. In this paper, we study a class of singular Finsler metrics defined by a Riemann metric α and 1-form β and characterize those which are respectively Douglasian and locally projectively flat in dimension $n \geq 3$ by some equations. Our study shows that the main class induced is an m-Kropina metric plus a linear part on β. For this class with $m \neq -1$, the local structure of projectively flat case is determined, and it is proved that a Douglas m-Kropina metric must be Berwaldian and a projectively flat m-Kropina metric must be locally Minkowskian. It indicates that the singular case is quite different from the regular one.

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1 Introduction

There are two important projective invariants in projective Finsler geometry: the Douglas curvature ($\mathbf{D}$) and the Weyl curvature ($\mathbf{W}^o$ in dimension two and $\mathbf{W}$ in higher dimensions) ([4]). A Finsler metric is called *Douglasian* if $\mathbf{D} = 0$. Roughly speaking, a Douglas metric is a Finsler metric having the same geodesics as a Riemannian metric. A Finsler metric is said to be *locally projectively flat* if at every point, there are local coordinate systems in which geodesics are straight. As we know, the locally projectively flat class of Riemannian metrics is very limited, nothing but the class of constant sectional curvature (Beltrami Theorem). However, the class of locally projectively flat Finsler metrics is very rich. Douglas metrics form a rich class of Finsler metrics including locally projectively flat Finsler metrics, and meanwhile there are many Douglas metrics which are not locally projectively flat.

In this paper, we will concentrate on a special class of Finsler metrics: (α, β)-metrics, and characterize those which are Douglasian and locally projectively flat under the condition (2) below. An (α, β)-*metric* is defined by a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a 1-form $\beta = b_i(x)y^i$ on a manifold M, which can be expressed in the following form:

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where $\phi(s)$ is a function satisfying certain conditions. It is known that F is a regular Finsler metric if β satisfies $\|\beta\|_\alpha < b_o$ and $\phi(s)$ is C^∞ on $(-b_o, b_o)$ satisfying

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (\rho^2 - s^2)\phi''(s) > 0, \quad (|s| \leq \rho < b_o), \tag{1}$$

where b_o is a positive constant ([10]). If $\phi(0)$ is not defined or ϕ does not satisfy (1), then the (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ is singular. Singular Finsler metrics have a lot of applications in the real world ([1] [2]). Z. Shen also introduces singular Finsler metrics in [11].

Assume $\phi(s)$ is in the following form

$$\phi(s) := cs + s^m \varphi(s), \tag{2}$$

where c, m are constant with $m \neq 0, 1$ and $\varphi(s)$ is a C^∞ function on a neighborhood of $s = 0$ with $\varphi(0) = 1$, and further for convenience we put $c = 0$ if m is a negative integer. If $m = 0$, we have $\phi(0) = 1$ and this case appears in a lot of literatures. When $m \geq 2$ is an integer, (2) is equivalent to the following condition

$$\phi(0) = 0, \quad \phi^{(k)}(0) = 0 \ \ (2 \leq k \leq m-1), \quad \phi^{(m)}(0) = m!.$$

Another interesting case is $c = 0$ and $\varphi(s) \equiv 1$ in (2), and in this case, $F = \alpha\phi(s)$ is called an m-Kropina metric, and in particular a Kropina metric when $m = -1$.

The case $\phi(0) = 1$ has been studied in a lot of interesting research papers ([5]–[7] [9] [10], [14]–[16]). In [5] [9], the authors respectively study and characterize Douglas (α, β)-metrics and locally projectively flat (α, β)-metrics in dimension $n \geq 3$ and $\phi(0) = 1$, and further, the present author solves the case $n = 2$ and shows that the two-dimensional case is quite different from the higher dimensional ones ([16]). In singular case, there are some papers on the studies of m-Kropina metrics and Kropina metrics ([8] [12] [13] [20]). Further, in [17], the present author classifies a class of two-dimensional singular (α, β)-metrics $F = \alpha\phi(\beta/\alpha)$ with $\phi(s)$ satisfying the condition (2) which are Douglasian and locally projectively flat respectively. In this paper we will solve the singular case under the condition (2) in higher dimensions, which shows that the singular case is quite different form the regular condition $\phi(0) = 1$ (cf. [5] [9]).

Theorem 1.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an n-dimensional (α, β)-metric on an open subset $U \subset R^n$ $(n \geq 3)$, where ϕ satisfies (2). Suppose $db \neq 0$ in U and that β is not parallel with respect to α. If F is a Douglas metric, or locally projectively flat, then F must be in the following form*

$$F = c\bar{\beta} + \bar{\beta}^m \bar{\alpha}^{1-m}, \quad (\bar{\alpha} := \sqrt{\alpha^2 + k\beta^2}, \ \bar{\beta} := \beta), \tag{3}$$

where c, k are constant. Note that $\bar{\alpha}$ is Riemannian if $k > -1/b^2$.

If $b = constant$ in Theorem 1.1, there are other classes for the metric F (see Theorem 4.1 and Theorem 5.1 below). Theorem 1.1 also holds if $n = 2$, but there is much difference between $n = 2$ and $n \geq 3$ when we determine the local structures of F in (3) which is Douglasian or locally projectively flat (cf. [17]).

Theorem 1.1 naturally induces an important class of singular Finsler metric—m-Kropina metric $F = \beta^m \alpha^{1-m}$. When $m = -1$, $F = \alpha^2/\beta$ is called a Kropina metric. There have been some research papers on Kropina metrics ([8] [13] [20]). In [12], the present author and Z. Shen characterize m-Kropina metrics which are weakly Einsteinian.

Next we determine the local structure of the metric $F = c\beta + \beta^m \alpha^{1-m}$ which are Douglasian and locally projectively flat respectivley when $m \neq -1$. The method is the application of the following deformation on α and β which is defined by

$$\widetilde{\alpha} := b^m \alpha, \quad \widetilde{\beta} := b^{m-1} \beta. \tag{4}$$

The deformation (4) first appears in [12] for the research on weakly Einstein m-Kropina metrics. It also appears in [17]. It is very useful for m-Kropina metrics. Obviously, if F is an m-Kropina metric, then F keeps formally unchanged, namely,

$$F = \beta^m \alpha^{1-m} = \widetilde{\beta}^m \widetilde{\alpha}^{1-m}.$$

Further, $\widetilde{\beta}$ has unit length with respect to $\widetilde{\alpha}$, that is, $||\widetilde{\beta}||_{\widetilde{\alpha}} = 1$.

Theorem 1.2 *Let $F = c\beta + \beta^m\alpha^{1-m}$ be an $n(\geq 3)$-dimensional Douglas (α,β)-metric, where c, m are constant with $m \neq 0, \pm 1$. Then we have the following cases:*

(i) *($c = 0$) F can be written as $F = \widetilde{\alpha}^{1-m}\widetilde{\beta}^m$, where $\widetilde{\beta}$ is parallel with respect to the Riemann metric $\widetilde{\alpha}$, and further α, β are related with $\widetilde{\alpha}, \widetilde{\beta}$ by*

$$\alpha = \eta^{\frac{m}{m-1}}\widetilde{\alpha}, \quad \beta = \eta\widetilde{\beta}, \tag{5}$$

where $\eta = \eta(x) > 0$ is a scalar function. Further, F is actually Berwaldian.

(ii) *($c \neq 0$) F can be written as $F = c\eta\widetilde{\beta} + \widetilde{\beta}^m\widetilde{\alpha}^{1-m}$, where $\widetilde{\beta}$ is parallel with respect to the Riemann metric $\widetilde{\alpha}$ with $\eta\widetilde{\beta}$ being closed. Furhter we have (5).*

Theorem 1.3 *Let $F = c\beta + \beta^m\alpha^{1-m}$ be an $n(\geq 3)$-dimensional locally projectively flat (α,β)-metric, where c, m are constant with $m \neq 0, \pm 1$. Then we have the following cases:*

(i) *($c = 0$) F can be written as $F = \widetilde{\alpha}^{1-m}\widetilde{\beta}^m$, where $\widetilde{\alpha}$ is flat and $\widetilde{\beta}$ is parallel with respect to $\widetilde{\alpha}$, and thus $\widetilde{\alpha}$ and $\widetilde{\beta}$ can be locally written as*

$$\widetilde{\alpha} = |y|, \quad \widetilde{\beta} = y^1. \tag{6}$$

Further α, β are related with $\widetilde{\alpha}, \widetilde{\beta}$ by (5). Moreover F is locally Minkowskian.

(ii) *($c \neq 0$) F can be written as $F = c\eta\widetilde{\beta} + \widetilde{\beta}^m\widetilde{\alpha}^{1-m}$, where (6) and (5) hold with $\eta = \eta(x^1) > 0$. In this case, F is Berwaldian, or locally Minkowskian if and only if $c = 0$ or $\eta =$ constant in (5); and here $\eta =$ constant implies α is flat and β is parallel.*

For the two-dimensional case, we have proved that the metric $F = c\beta + \alpha^2/\beta$ is always Douglasian, the m-Kropina metric in Theorem 1.2(i) is locally Minkowskian (determined by Theorem 1.3(i)), and the metric F in Theorem 1.2(ii) is locally projectively flat if additionally $m \neq -3$ ([17]). When $\widetilde{\alpha}$ is Not flat and $\widetilde{\beta}$ is parallel with respect to $\widetilde{\alpha}$, then the m-Kropina metric F in Theorem 1.2(i) is Douglasian but Not locallly projectively flat, and a family of concrete examples to this case are given in the last section.

When $m = -1$, the deformation (4) cannot be applied to Theorem 1.2 and 1.3 to determine the local structure of $F = c\beta + \alpha^2/\beta$ which is Douglasian or locallly projectively flat. See the general characterization in Theorem 6.1 and 6.2 respectively below. In [18], we further prove that for the dimensions $n \geq 2$, if $F = c\beta + \alpha^2/\beta$ is locally projectively flat with constant flag curvature, then F is locally Minkowskian. If $c\beta$ is small, then $F = (\alpha^2 + c\beta^2)/\beta = \bar{\alpha}^2/\beta$ is a Kropina metric. In [19], the present author has shown some non-trivial examples of Kropina metrics which are locally projectively flat.

Open Problem: Determine the local structure of the $n(\geq 3)$-dimensional metric $F = c\beta + \alpha^2/\beta$ which is Douglasian or locallly projectively flat.

2 Preliminaries

Let $F = F(x,y)$ be a Finsler metric on an n-dimensional manifold M. In local coordinates, the spray coefficients G^i are defined by

$$G^i := \frac{1}{4}g^{il}\big\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\big\}. \tag{7}$$

If F is a Douglas metric, then G^i are in the following form:

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k + P(x,y)y^i, \tag{8}$$

where $\Gamma^i_{jk}(x)$ are local functions on M and $P(x,y)$ is a local positively homogeneous function of degree one in y. It is easy to see that F is a Douglas metric if and only if $G^iy^j - G^jy^i$ is a homogeneous polynomial in (y^i) of degree three, which by (8) can be written as ([3]),

$$G^iy^j - G^jy^i = \frac{1}{2}(\Gamma^i_{kl}y^j - \Gamma^j_{kl}y^i)y^ky^l.$$

According to G. Hamel's result, a Finsler metric F is projectively flat in U if and only if

$$F_{x^my^l}y^m - F_{x^l} = 0. \tag{9}$$

The above formula implies that $G^i = Py^i$ with P given by

$$P = \frac{F_{x^m}y^m}{2F}. \tag{10}$$

Consider an (α,β)-metric $F = \alpha\phi(\beta/\alpha)$. The spray coefficients G^i_α of α are given by

$$G^i_\alpha = \frac{1}{4}a^{il}\big\{[\alpha^2]_{x^ky^l}y^k - [\alpha^2]_{x^l}\big\}.$$

Let $\nabla\beta = b_{i|j}y^idx^j$ denote the covariant derivatives of β with respect to α and define

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_j := b^ir_{ij}, \quad s_j := b^is_{ij}, \quad s^i := a^{ik}s_k,$$

where $b^i := a^{ij}b_j$ and (a^{ij}) is the inverse of (a_{ij}). By (7) again, the spray coefficients G^i of F are given by:

$$G^i = G^i_\alpha + \alpha Qs^i_0 + \alpha^{-1}\Theta(-2\alpha Qs_0 + r_{00})y^i + \Psi(-2\alpha Qs_0 + r_{00})b^i, \tag{11}$$

where $s^i_j = a^{ik}s_{kj}, s^i_0 = s^i_ky^k, s_i = b^ks_{ki}, s_0 = s_iy^i$, and

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \frac{Q - sQ'}{2\Delta}, \quad \Psi := \frac{Q'}{2\Delta}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q'.$$

By (11) one can see that $F = \alpha\phi(\beta/\alpha)$ is a Douglas metric if and only if

$$\alpha Q(s^i_0y^j - s^j_0y^i) + \Psi(-2\alpha Qs_0 + r_{00})(b^iy^j - b^jy^i) = \frac{1}{2}(G^i_{kl}y^j - G^j_{kl}y^i)y^ky^l, \tag{12}$$

where $G^i_{kl} := \Gamma^i_{kl} - \gamma^i_{kl}$, Γ^i_{kl} are given in (8) and $\gamma^i_{kl} := \partial^2G^i_\alpha/\partial y^k\partial y^l$.

Further, $F = \alpha\phi(\beta/\alpha)$ is projectively flat on $U \subset R^n$ if and only if

$$(a_{ml}\alpha^2 - y_my_l)G^m_\alpha + \alpha^3Qs_{l0} + \Psi\alpha(-2\alpha Qs_0 + r_{00})(\alpha b_l - sy_l) = 0, \tag{13}$$

where $y_l = a_{ml}y^m$.

The following lemma is obvious.

Lemma 2.1 *If $Q = ks$, where k is a constant, then $\phi(s) = c\sqrt{1 + ks^2}$ for some constant c.*

3 Equations in a Special Coordinate System

Fix an arbitrary point $x \in M$ and take an orthogonal basis $\{e_i\}$ at x such that

$$\alpha = \sqrt{\sum_{i=1}^{n}(y^i)^2}, \quad \beta = by^1.$$

Then we change coordinates (y^i) to (s, y^a) such that

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha},$$

where $\bar{\alpha} = \sqrt{\sum_{a=2}^{n}(y^a)^2}$. Let

$$\bar{r}_{10} := r_{1a}y^a, \quad \bar{r}_{00} := r_{ab}y^a y^b, \quad \bar{s}_0 := s_a y^a.$$

We have $\bar{s}_0 = b\bar{s}_{10}, s_1 = bs_{11} = 0$. In the following, we also put

$$\bar{G}^0_{10} := G^a_{1b}y_a y^b, \ \bar{G}^1_{00} := G^1_{ab}y^a y^b, \ \bar{G}^a_{00} := G^a_{bc}y^b y^c, \ etc.$$

Then by the above coordinate (s, y^a) and using (12) and (13), it follows from [5] [9] we have the following lemmas:

Lemma 3.1 ([5]) *For $n \geq 2$, an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ is a Douglas metric if and only if there hold the following four identities:*

$$\frac{bQ\bar{s}^a_0 s - \Psi r_{11}s^2 by^a}{b^2 - s^2}\bar{\alpha}^2 - \Psi\bar{r}_{00}by^a = \frac{s^2}{2(b^2 - s^2)}(\bar{G}^a_{10} + \bar{G}^a_{01} - G^1_{11}y^a)\bar{\alpha}^2 - \frac{1}{2}\bar{G}^1_{00}y^a, \tag{14}$$

$$\frac{bQs^2 s^a_1}{b^2 - s^2}\bar{\alpha}^2 + (-2\Psi s\bar{r}_{10} + 2\Psi Qb^2 s^1_0 - Qs^1_0)by^a = \frac{G^a_{11}s^3}{2(b^2 - s^2)}\bar{\alpha}^2 + \frac{1}{2}\{\bar{G}^a_{00} - (\bar{G}^1_{10} + \bar{G}^1_{01})y^a\}s, \tag{15}$$

$$\frac{bs}{b^2 - s^2}(s^a_1 y^b - s^b_1 y^a)Q\bar{\alpha}^2 = \frac{s^2}{2(b^2 - s^2)}(G^a_{11}y^b - G^b_{11}y^a)\bar{\alpha}^2 + \frac{1}{2}(\bar{G}^a_{00}y^b - \bar{G}^b_{00}y^a), \tag{16}$$

$$(\bar{s}^a_0 y^b - \bar{s}^b_0 y^a)bQ = \frac{s}{2}\{(\bar{G}^a_{10} + \bar{G}^a_{01})y^b - (\bar{G}^b_{10} + \bar{G}^b_{01})y^a\}. \tag{17}$$

Lemma 3.2 ([5] [9]) *$(n \geq 2)$ Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric. Suppose Ψ is dependent on s, β is not parallel with respect to α and β is closed. Then F is a Douglas metric if and only if*

$$b_{i|j} = 2\bar{\tau}\{\delta b_i b_j + \eta(b^2 a_{ij} - b_i b_j)\}. \tag{18}$$

$$2\Psi = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\delta s^2 + \eta(b^2 - s^2)}, \tag{19}$$

where $\bar{\tau} = \bar{\tau}(x), \lambda = \lambda(x), \mu = \mu(x), \delta = \delta(x), \eta = \eta(x)$ are scalar functions satisfying $\lambda\eta - \mu\delta \neq 0$. F is projectively flat if and only if (18), (19) and

$$G^i_\alpha = \rho y^i - \bar{\tau}\{\lambda\beta^2 + \mu(b^2\alpha^2 - \beta^2)\}b^i \tag{20}$$

hold, where $\rho := \rho_i(x)y^i$ is a 1-form.

Lemma 3.3 ([9]) *For $n \geq 2$, if $s_{ab} = 0$, then an (α,β)-metric $F = \alpha\phi(\beta/\alpha)$ is locally projectively flat if and only if*

$$\begin{aligned} 0 &= \bar{G}^a_{10}\bar{\alpha}^2 - \bar{G}^0_{10}y^a, && (21)\\ 0 &= (\bar{r}_{00} + \frac{s^2 r_{11}\bar{\alpha}^2}{b^2-s^2})b\Psi - \frac{s^2}{2(b^2-s^2)}(2\bar{G}^0_{10} - G^1_{11}\bar{\alpha}^2) + \frac{1}{2}\bar{G}^1_{00}, && (22)\\ 0 &= \bar{G}^a_{00} - \Big\{\frac{2bQ(1-2b^2\Psi)\bar{s}_{10}}{s} + 4b\Psi\bar{r}_{10} + 2\bar{G}^1_{10}\Big\}y^a + \frac{s(G^a_{11}s - 2bQs_{1a})\bar{\alpha}^2}{b^2-s^2}. && (23) \end{aligned}$$

where G^i_{jk} are the spray coefficients of α.

Note that in (16), if $\lim_{s\to 0} sQ = 0$ and Q/s is dependent on s, then we can get $s^a_1 = 0$. The zero limit is a key factor to prove β is closed using (16) and (17). In singular case, we generally don't have $\lim_{s\to 0} sQ = 0$.

4 Douglas (α,β)-metrics

In this section, we characterize a class of $n(\geq 3)$-dimensional singular (α,β)-metrics which are Douglas metrics. We have the following theorem.

Theorem 4.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $n(\geq 3)$-dimensional (α,β)-metric on an open subset $U \subset R^n$, where ϕ satisfies (2). Suppose that β is not parallel with respect to α. Then F is a Douglas metric if and only if one of the following cases holds:*

(i) *ϕ and β satisfy*

$$\phi(s) = cs + \frac{1}{s}, \quad s_{ij} = \frac{b_i s_j - b_j s_i}{b^2}, \tag{24}$$

where c is a constant.

(ii) *ϕ and β satisfy*

$$\begin{aligned} \phi(s) &= k_1 s + s^m(1+k_2 s^2)^{\frac{1-m}{2}}, && (25)\\ b_{i|j} &= 2\tau\big\{mb^2 a_{ij} - (m+1+k_2b^2)b_ib_j\big\}, && (26) \end{aligned}$$

where $\tau = \tau(x)$ is a scalar function and k_1, k_2 are constant.

(iii) *ϕ and β satisfy*

$$\begin{aligned} \phi(s) &= s^m(1+ks^2)^{\frac{1-m}{2}}, && (27)\\ r_{ij} &= 2\tau\big\{mb^2a_{ij} - (m+1+kb^2)b_ib_j\big\} - \frac{m+1+2kb^2}{(m-1)b^2}(b_is_j + b_js_i), && (28)\\ s_{ij} &= \frac{b_is_j - b_js_i}{b^2}, && (29) \end{aligned}$$

where k is constant and $\tau = \tau(x)$ is a scalar.

In Theorem 4.1 (iii), if $b = constant$, then $k = -1/b^2$ in (27)–(28), and we get

$$\begin{aligned} \phi(s) &= s^m\big\{1 - (\frac{s}{b})^2\big\}^{\frac{1-m}{2}}, && (30)\\ r_{ij} &= 2\bar{\tau}(b^2a_{ij} - b_ib_j) - \frac{1}{b^2}(b_is_j + b_js_i), && (31) \end{aligned}$$

where $\bar{\tau} := m\tau$. Note that if $n = 2$, (31) is equivalent to $b = constant$ (see [7]), and clearly (29) holds automatically.

4.1 $d\beta = 0$

Assume ϕ satisfies (2), β is not parallel with respect to α and β is closed. Obviously F is not of Randers type. So by Lemma 3.2 we have (19). We first determine $\lambda, \eta, \delta, \mu$ in (19). Rewrite (19) as follows

$$[\delta s^2 + \eta(b^2 - s^2)]\phi'' = [\lambda s^2 + \mu(b^2 - s^2)][\phi - s\phi' + (b^2 - s^2)\phi'']. \tag{32}$$

Plug

$$\phi(s) = a_1 s + s^m(1 + a_{m+1}s + a_{m+2}s^2 + a_{m+3}s^3 + a_{m+4}s^4) + o(s^{m+4})$$

into (32). Let p_i be the coefficients of s^i in (32). First $p_{m-2} = 0$ gives

$$\eta = \mu b^2. \tag{33}$$

Plugging (33) into $p_m = 0$ yields

$$\delta = \lambda b^2 - \frac{m+1}{m}\mu b^2. \tag{34}$$

Case A. Assume $m = -1$. Plug (33), (34) and $m = -1$ into (32) and then we get

$$s^2\phi'' + s\phi' - \phi = 0,$$

whose solution is given by (24).
Case B. Assume $m \neq -1$. Plugging (33) and (34) into $p_{m+2} = 0$ yields

$$\lambda = [m(m-1) + 2a_{m+2}b^2]\epsilon, \quad \mu = m(m-1)\epsilon, \tag{35}$$

where $\epsilon = \epsilon(x) \neq 0$ is a scalar. It is easy to see that

$$\lambda\eta - \mu\delta = m(m+1)(m-1)^2 b^2 \epsilon^2 \neq 0. \tag{36}$$

Plug (33), (34) and (35) into (19) and we get

$$2\Psi = \frac{\phi''}{\phi - s\phi' + (b^2 - s^2)\phi''} = \frac{m(m-1) + 2a_{m+2}s^2}{m(m-1)b^2 + (1 - m^2 + 2a_{m+2}b^2)s^2}, \tag{37}$$

which can be rewritten as

$$\phi'' = \frac{-m + k_2 s^2}{(1 + k_2 s^2)s^2}(\phi - s\phi'), \tag{38}$$

where we put

$$k_1 = a_1, \quad k_2 = -2a_{m+2}/(m-1).$$

Solving the differential equation (38) gives (25). Plug (33), (34) and (35) into (18) and we get (26), where we put

$$\tau := (m-1)\epsilon b^2 \bar{\tau}. \tag{39}$$

4.2 $d\beta \neq 0$

We will deal with the equations (14)–(17) respectively.

Step (1): By Lemma 2.1 and the assumption on ϕ, we see Q/s is dependent on s. So by (17), we have

$$\bar{s}^a_0 y^b - \bar{s}^b_0 y^a = 0,$$

from which we have $s_{ab}=0$ since $n\geq 3$. Therefore, we obtain (29).

Step (2): We rewrite (16) as

$$0=s\big[2b\phi' s_{1a}+s(\phi-s\phi')G^a_{11}\big]y^b-s\big[2b\phi' s_{1b}+s(\phi-s\phi')G^b_{11}\big]y^a+(b^2-s^2)(\phi-s\phi')\theta_{ab}, \quad (40)$$

where θ_{ab} are defined by

$$\theta_{ab}:=\bar{G}^a_{00}y^b-\bar{G}^b_{00}y^a.$$

Plug

$$\phi(s)=a_1 s+s^m(1+a_{m+1}s+a_{m+2}s^2+a_{m+3}s^3+o(s^{m+3})$$

into (40). Let p_i denote the coefficient of s^i in (40). By $p_m=0$ we get

$$\theta_{ab}=\frac{2m(s_{1a}y^b-s_{1b}y^a)}{(m-1)b}. \tag{41}$$

Substituting (41) into $p_{m+2}=0$ yields

$$T_b y^a-T_a y^b=0, \tag{42}$$

where T_a are defined by

$$T_a:=(m-1)^2bG^a_{11}+2(m-m^2+2a_{m+2}b^2)s_{1a}.$$

Since $n\geq 3$, by (42) we have $T_a=0$, which are written as

$$G^a_{11}=-\frac{2(m-m^2+2a_{m+2}b^2)}{(m-1)^2b}s_{1a}. \tag{43}$$

Finally, plug (41) and (43) into (16) and then we obtain

$$Q=-\frac{m+ks^2}{(m-1)s}, \tag{44}$$

where we have used the fact that $s_{1a}y^b-s_{1b}y^a\neq 0$ since β is not closed and $s_{ab}=0$, and k is defined by

$$k:=-\frac{2a_{m+2}}{m-1},$$

Solving the ODE (44) we get $\phi(s)$ given by (27).

Step (3): Plug (43) and (44) into (15) and then we get

$$A_0s^2+mb^2A_1=0,$$

where A_0, A_1 are polynomials in (y^a) independent of s. By $A_1=0$ we get

$$\bar{G}^a_{00}=\Big\{\bar{G}^1_{10}+\bar{G}^1_{01}-\frac{2\bar{r}_{10}}{b}-\frac{2(m+1)\bar{s}_{10}}{(m-1)b}\Big\}y^a+\frac{2ms_{1a}\bar{\alpha}^2}{(m-1)b}. \tag{45}$$

Plugging (45) into $A_0=0$ gives

$$2(m+1)\big[(m-1)\bar{r}_{10}+(m+1+2kb^2)\bar{s}_{10}\big]y^a=0.$$

So if $m \neq -1$ we get

$$\bar{r}_{10} = -\frac{m+1+2kb^2}{m-1}\bar{s}_{10}. \tag{46}$$

Now we show $m - kb^2 \neq 0$ if $m \neq -1$, which will be needed in the following. If $m - kb^2 = 0$, then $b = \mathit{constant}$, and by (46) get

$$0 = \bar{r}_{10} + \bar{s}_{10} = -\frac{(m+1)\bar{s}_{10}}{m-1},$$

which is impossible since $\bar{s}_{10} \neq 0$.

Step (4): By $s_{ab} = 0$ and a simple analysis on (14), we see (14) can be written as

$$\frac{s^2}{2(b^2-s^2)}(G^1_{11} - \gamma)\delta_{ab} + \frac{1}{4}(G^1_{ab} + G^1_{ba}) = b\Psi\left(\frac{r_{11}s^2}{b^2-s^2}\delta_{ab} + r_{ab}\right), \tag{47}$$

where $\gamma := G^a_{1a} + G^a_{a1}$ (not summed) which is independent of the index a. By (44) and the definition of Ψ we have

$$\Psi = \frac{ks^2 - m}{2\big[(1+m+kb^2)s^2 - mb^2\big]}. \tag{48}$$

Plug $s_{ab} = 0$ and (48) into (47) and we obtain

$$B_0 s^4 + bB_1 s^2 + mb^3 B_2 = 0,$$

where B_0, B_1, B_2 are scalar functions independent of s. Then by $B_2 = 0$ we have

$$G^1_{ba} = \frac{2r_{ab}}{b} - G^1_{ab}. \tag{49}$$

If $m \neq -1$, using $m - kb^2 \neq 0$, plug (49) into $B_0 = 0, B_1 = 0$ and then we obtain

$$r_{11} = \frac{b(1+kb^2)(\gamma - G^1_{11})}{m - kb^2}, \quad r_{ab} = \frac{mb(\gamma - G^1_{11})}{m - kb^2}\delta_{ab}. \tag{50}$$

Now summed up from the above, it follows from (46) and (50) that (28) holds if $m \neq -1$, where τ is defined by

$$\tau := \frac{G^1_{11} - \gamma}{2b(m - kb^2)}.$$

4.3 The inverse of the case $m = -1$

We have shown that if $F = c\beta + \alpha^2/\beta$ is a Douglas metric, then s_{ij} are given by (24). There are different ways to show the inverse is also true. We prove the inverse by (12). We only need to show the left hand side of (12) are polynomials in y of degree three.

Plug

$$\phi(s) = cs + \frac{1}{s}, \quad s^i_0 = \frac{b^i s_0 - \beta s^i}{b^2}$$

into the left hand side of (12), and then we get

$$\frac{1}{2b^2}\big\{(y^j s^i - y^i s^j)(\alpha^2 - c\beta^2) + (b^i y^j - b^j y^i) r_{00}\big\},$$

which are clearly polynomials in y of degree three.

5 Projectively flat (α, β)-metrics

In this section, we characterize a class of $n(\geq 3)$-dimensional singular (α, β)-metrics which are projectively flat. We have the following theorem.

Theorem 5.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $n(\geq 3)$-dimensional (α, β)-metric on an open subset $U \subset R^n$, where ϕ satisfies (2). Suppose that β is not parallel with respect to α. Let G^i_α be the spray coefficients of α. Then F is projectively flat in U with $G^i = P(x, y)y^i$ if and only if one of the following cases holds:*

(i) *$\phi(s)$ and β satisfy (24), and G^i_α satisfy*

$$G^i_\alpha = \rho y^i - \frac{r_{00}}{2b^2} b^i - \frac{\alpha^2 - c\beta^2}{2b^2} s^i. \tag{51}$$

In this case, the projective factor P is given by

$$P = \rho - \frac{1}{b^2(\alpha^2 + c\beta^2)} \Big\{ (\alpha^2 - c\beta^2)s_0 + r_{00}\beta \Big\}. \tag{52}$$

(ii) *$\phi(s)$ and β satisfy (25) and (26), and G^i_α satisfy*

$$G^i_\alpha = \rho y^i - \tau(m\alpha^2 - k_2\beta^2)b^i. \tag{53}$$

In this case, the projective factor P is given by

$$P = \rho + \tau\alpha \Big\{ s(-m + k_2 s^2) - s^2(1 + k_2 s^2)\frac{\phi'}{\phi} \Big\}. \tag{54}$$

(iii) *$\phi(s)$ and β satisfy (27)–(29), and G^i_α satisfy*

$$G^i_\alpha = \rho y^i + \Big\{ \frac{2k\beta s_0}{(m-1)b^2} - \tau(m\alpha^2 - k\beta^2) \Big\} b^i - \frac{m\alpha^2 + k\beta^2}{(m-1)b^2} s^i. \tag{55}$$

In this case, the projective factor P is given by

$$P = \rho - 2m\tau\beta - \frac{2m}{(m-1)b^2} s_0. \tag{56}$$

The above function $\rho = \rho_i(x)y^i$ is a 1-form.

Proof : Our proof of Theorem 5.1 breaks into two cases: $m = -1$ and $m \neq -1$. Firstly by (21) we have

$$G^a_{1b} = \frac{\xi}{2}\delta_{ab}, \tag{57}$$

where $\xi = \xi(x)$ is a scalar function.

Step 1. Assume $m = -1$.

Plug $\phi(s) = cs + 1/s$ into (23) and we get

$$G^a_{11} = -\frac{1 - kb^2}{b} s_{1a}, \quad \bar{G}^a_{00} = (\frac{2\bar{r}_{10}}{b} + 2\bar{G}^1_{10})y^a - \frac{\bar{\alpha}^2}{b} s_{1a}. \tag{58}$$

Next substitute $\phi(s) = cs + 1/s$ and (57) into (22) and we have

$$\bar{G}^1_{00} = -\frac{\bar{r}_{00}}{b}, \quad G^1_{11} = \xi - \frac{r_{11}}{b}. \tag{59}$$

Now by (58) and (59) we get (51), where ρ is defined by

$$\rho := \frac{1}{2}\xi y^1 + (\frac{r_{1a}}{b} + G^1_{1a})y^a.$$

Finally, we solve the projective factor. Plug $\phi(s) = cs+1/s$, (51) and $s^i_0 = (b^i s_0 - \beta s^i)/b^2$ into (11), and then we get $G^i = Py^i$ with P given by (52).

Step 2. Assume $m \neq -1$.
Case A: Assume $d\beta = 0$. Plugging (35) into (20) gives (53). Next we show (54). By (26) we have

$$r_{00} = 2\tau\big\{mb^2\alpha^2 - (1 + m + k_2b^2)\beta^2\big\}. \tag{60}$$

Now plug $s_{i0} = 0, s_0 = 0$ and (53), (38) and (60) into (11), and then we obtain (54).

Case B: Assume $d\beta \neq 0$. Plugging (44) and (46) into (23) gives

$$\bar{G}^a_{00} = \big(2\bar{G}^1_{10} - \frac{4kb\bar{s}_{10}}{m-1}\big)y^a - \frac{2m\bar{\alpha}^2 s_{1a}}{(m-1)b}, \quad G^a_{11} = -\frac{2(m+kb^2)s_{1a}}{(m-1)b}. \tag{61}$$

Next substituting (48), (50) and (57) into (22) gives

$$\bar{G}^1_{00} = -2mb\tau\bar{\alpha}^2, \quad G^1_{11} = \xi - 2b(m - kb^2)\tau. \tag{62}$$

Now by (61) and (62) we get (55), where ρ is defined by

$$\rho := \frac{1}{2}\xi y^1 + (-\frac{2kbs_{1a}}{m-1} + G^1_{1a})y^a.$$

Finally, we solve the projective factor. By (28) and (29) we have

$$r_{00} = 2\tau\big\{mb^2\alpha^2 - (1 + m + k_2b^2)\beta^2\big\} - 2\frac{m+1+2kb^2}{(m-1)b^2}\beta s_0, \quad s^i_0 = \frac{b^i s_0 - \beta s^i}{b^2}. \tag{63}$$

Now plug (44), (48), (55) and (63) into (11), and then we obtain (56).

6 Proof of Theorem 1.2 and Theorem 1.3

Based on Theorem 4.1 and Theorem 5.1, we give a general characterization for $F = c\beta + \beta^m\alpha^{1-m}$ to be Douglasian and locally projectively flat respectively.

Theorem 6.1 *Let $F = c\beta + \beta^m\alpha^{1-m}$ be an n-dimensional (α, β)-metric on an open subset $U \subset R^n$ $(n \geq 3)$, where c, m are constant with $m \neq 0, 1$. Then for some scalar function $\tau = \tau(x)$, we have the following cases:*

(i) $(m = -1)$ *F is a Douglas metric if and only if β satisfies*

$$s_{ij} = \frac{b_i s_j - b_j s_i}{b^2}. \tag{64}$$

(ii) $(c \neq 0, m \neq -1)$ *F is a Douglas metric if and only if β satisfies*

$$b_{i|j} = 2\tau\{mb^2 a_{ij} - (m+1)b_i b_j\}, \tag{65}$$

(iii) $(c = 0, m \neq -1)$ *F is a Douglas metric if and only if β satisfies (64) and*

$$r_{ij} = 2\tau\{mb^2 a_{ij} - (m+1)b_i b_j\} - \frac{m+1}{(m-1)b^2}(b_i s_j + b_j s_i), \tag{66}$$

Theorem 6.2 *Let $F = c\beta + \beta^m \alpha^{1-m}$ be an n-dimensional (α, β)-metric on an open subset $U \subset R^n$ $(n \geq 3)$, where c, m are constant with $m \neq 0, 1$. Then for some scalar function $\tau = \tau(x)$ and 1-form $\rho = \rho_i(x)y^i$, we have the following cases:*

(i) $(m = -1)$ *F is projectively flat if and only if β satisfies (64) and G^i_α satisfy*

$$G^i_\alpha = \rho y^i - \frac{r_{00}}{2b^2} b^i - \frac{\alpha^2 - c\beta^2}{2b^2} s^i. \tag{67}$$

(ii) $(c \neq 0; m \neq -1)$ *F is projectively flat if and only if β satisfies (65) and G^i_α satisfy*

$$G^i_\alpha = \rho y^i - m\tau\alpha^2 b^i. \tag{68}$$

(iii) $(c = 0; m \neq -1)$ *F is projectively flat if and only if β satisfies (64) and (66), and G^i_α satisfy*

$$G^i_\alpha = \rho y^i - m\tau\alpha^2 b^i + \frac{m}{(1-m)b^2}\alpha^2 s^i. \tag{69}$$

We can use the deformation (4) to simplify (64), (66) and (69), which is shown as follows:

Lemma 6.3 *For a pair (α, β), suppose β satisfies (66) and (64). Then under the deformation (4), $\widetilde{\beta}$ must be parallel with respect to $\widetilde{\alpha}$. Further, if the spray coefficients G^i_α of α satisfy (69), then $\widetilde{\alpha}$ is projectively flat.*

Proof : By (66) and (64), a direct computation under (4) gives $\widetilde{r}_{ij} = 0$ and $\widetilde{s}_{ij} = 0$ respectively. Thus $\widetilde{\beta}$ is parallel with respect to $\widetilde{\alpha}$. If (69) holds, then under (4) we have

$$\widetilde{G}^i_{\widetilde{\alpha}} = \Big[\rho - 2m\tau\beta - \frac{2ms_0}{(m-1)b^2}\Big] y^i.$$

So $\widetilde{\alpha}$ is projectively flat.

We can also give another simple proof for $m \neq -1$. Define $F := \beta^m \alpha^{1-m}$. If (66) and (64) hold, then F is a Douglas metric by Theorem 6.1(iii). Since F keeps formally unchanged under (4), by Theorem 6.1(iii) we have

$$\widetilde{r}_{ij} = 2\widetilde{\tau}\{m\widetilde{a}_{ij} - (m+1)\widetilde{b}_i\widetilde{b}_j\} - \frac{m+1}{m-1}(\widetilde{b}_i\widetilde{s}_j + \widetilde{b}_j\widetilde{s}_i), \tag{70}$$

$$\widetilde{s}_{ij} = \widetilde{b}_i\widetilde{s}_j - \widetilde{b}_j\widetilde{s}_i, \quad \widetilde{b}^2 = 1. \tag{71}$$

Contracting (70) by $\widetilde{b}^i$ and then by $\widetilde{b}^j$ and using $\widetilde{r}_i + \widetilde{s}_i = 0$, it is easy to get $\widetilde{\tau} = 0$, $\widetilde{r}_{ij} = 0$ and $\widetilde{s}_i = 0$. So by (71) we have $\widetilde{s}_{ij} = 0$. Further, if G^i_α satisfy (69), then F is locally

projectively flat by Theorem 6.2(iii). Again, since F keeps formally unchanged under (4), by Theorem 6.2(iii) and $\widetilde{b}=1$, and using $\widetilde{\tau}=0$ and $\widetilde{s}_i=0$, we have

$$\widetilde{G}^i_{\widetilde{\alpha}}=\widetilde{\rho}y^i-m\widetilde{\tau}\widetilde{\alpha}^2\widetilde{b}^i+\frac{m}{1-m}\widetilde{\alpha}^2\widetilde{s}^i=\widetilde{\rho}y^i,$$

which imply $\widetilde{\alpha}$ is projectively flat. Q.E.D.

Proof of Theorem 1.2 :

If $c=0$, then we have $F=\beta^m\alpha^{1-m}$. Since $F=\beta^m\alpha^{1-m}$ with $m\neq-1$ is a Douglas metric, by Theorem 6.1(iii) we have (66) and (64). Put $\eta:=||\beta||^{1-m}_{\alpha}$ and then we get (5). Then by Lemma 6.3 we complete the proof of Theorem 1.2(i). If $c\neq0$, then β is closed by Theorem 6.1(ii). Thus $\eta\widetilde{\beta}$ is closed. This completes the proof of Theorem 1.2(ii). Q.E.D.

Proof of Theorem 1.3 :

Case A: Assume $c=0$. Since $F=\beta^m\alpha^{1-m}$ is a Douglas metric, we have (66), (64) and (69) by Theorem 6.2(iii). Then Lemma 6.3 shows that under the deformation (4), $\widetilde{\beta}$ is parallel with respect to $\widetilde{\alpha}$, and $\widetilde{\alpha}$ is projectively flat. Thus we can first locally express $\widetilde{\alpha}$ in the following form

$$\widetilde{\alpha}=\frac{\sqrt{(1+\mu|x|^2)|y|^2-\mu\langle x,y\rangle^2}}{1+\mu|x|^2},\tag{72}$$

where μ is the constant sectional curvature of $\widetilde{\alpha}$. Since $\widetilde{\beta}=\widetilde{b}_iy^i$ is of course a closed 1-form which is conformal with respect to $\widetilde{\alpha}$, it has been shown in [21] the following

$$\widetilde{b}_i=\frac{kx^i+(1+\mu|x|^2)e_i-\mu\langle e,x\rangle x^i}{(1+\mu|x|^2)^{\frac{3}{2}}},\quad \widetilde{b}^i=\sqrt{1+\mu|x|^2}(kx^i+e_i).\tag{73}$$

where k is a constant and $e=(e_i)$ is a constant vector, and $\widetilde{b}_i=\widetilde{a}_{ij}\widetilde{b}^j$. By (73) we have

$$1=\widetilde{b}^2=||\widetilde{\beta}||^2_{\widetilde{\alpha}}=|e|^2+\frac{k^2|x|^2+2k\langle e,x\rangle-\mu\langle e,x\rangle^2}{1+\mu|x|^2}.\tag{74}$$

It is easy to conclude from (74) that $\mu=0$. So $\widetilde{\alpha}$ is flat. Thus $\widetilde{\alpha}$ and $\widetilde{\beta}$ can be locally expressed as (6).

Case B: Assume $c\neq0$. In this case, we only need to require additionally that β be closed by Theorem 6.2(ii). Then since $\beta=\eta\widetilde{\beta}=\eta y^1$ is closed, we see $\eta=\eta(x^1)$. Now we can easily verify that for the metric $F=c\eta\widetilde{\beta}+\widetilde{\beta}^m\widetilde{\alpha}^{1-m}$, (9) holds. So F is projectively flat with $G^i=Py^i$. Further, by (10) we can get the projective factor P given by

$$P=\frac{c\eta_1}{2F}(y^1)^2,\quad \eta_1:=\eta_{x^1},\tag{75}$$

and the scalar flag curvature K is given by

$$K=\frac{c(y^1)^3}{2F^3}\Big\{\frac{3c\eta_1^2y^1}{2F}-\eta_{11}\Big\},\quad \eta_{11}:=\eta_{x^1x^1}.\tag{76}$$

Then by (75) and (76), F is Berwaldian, or locally Minkowskian if and only if $c=0$ or $\eta=$ *constant*.

7 A local representation

We have show that if $\widetilde{\alpha}$ is Not flat and $\widetilde{\beta}$ is parallel with respect to $\widetilde{\alpha}$, then the m-Kropina metric F in Theorem 1.2(i) is Douglasian but Not locallly projectively flat. In this section, we give a family of examples to this case.

Firstly we show a lemma based on [14] (also see [15]).

Lemma 7.1 *Let $\widetilde{\alpha}$ be an n-dimensional Riemann metric which is locally conformally flat, and $\widetilde{\beta}$ is a 1-form. Then $\widetilde{\beta}$ is a Killing form $\widetilde{r}_{ij}=0$ with unit length if and only if $\widetilde{\alpha}$ and $\widetilde{\beta}$ can be locally expressed as*

$$\widetilde{\alpha}=\sqrt{\frac{|y|^2}{|u|^2}},\quad \widetilde{\beta}=\frac{\langle u,y\rangle}{|u|^2},\tag{77}$$

where $u:=(u^1(x),\cdots,u^n(x))$ is a vector satisfying the following PDEs (fixed i,j):

$$\frac{\partial u^i}{\partial x^j}+\frac{\partial u^j}{\partial x^i}=0\ (\forall\, i\neq j),\qquad \frac{\partial u^i}{\partial x^i}=\frac{\partial u^j}{\partial x^j}\ (\forall\, i,j).\tag{78}$$

Further, if $n=2$, $\widetilde{\beta}$ is parallel with respect to $\widetilde{\alpha}$, and $\widetilde{\alpha}$ is flat.

Proof : Suppose $\widetilde{\beta}$ satisfies $\widetilde{r}_{ij}=0$ and has unit length. Since $\widetilde{\alpha}$ is locally conformally flat, we can express it as $\widetilde{\alpha}=e^{\frac{1}{2}\sigma(x)}|y|$. In this case, firstly we can express $\widetilde{\beta}=e^{\sigma}\langle u,y\rangle$, and then by [14], u satisfies (78). Since $\widetilde{\beta}$ has unit length, clearly we have $e^{\sigma}=1/|u|^2$. So we get (77).

Conversely, suppose $\widetilde{\alpha}$ and $\widetilde{\beta}$ are given by (77) with u satisfying (78). Clearly $\widetilde{\beta}$ has unit length. Next we verify $\widetilde{r}_{ij}=0$. It has been shown in [14] that if $\widetilde{\alpha}$ and $\widetilde{\beta}$ are given by (77) with u satisfying (78), then $\widetilde{\beta}$ is a conformal form satisfying

$$\widetilde{r}_{ij}=\frac{\partial u^1}{\partial x^1}+\frac{1}{2}u^k\sigma_k,\tag{79}$$

where

$$\sigma:=-\ln(|u|^2),\quad \sigma_k:=\sigma_{x^k}.$$

Then (79) becomes

$$\widetilde{r}_{ij}=\frac{1}{|u|^2}\big(\frac{\partial u^1}{\partial x^1}|u|^2-u^iu^k\frac{\partial u^i}{\partial x^k}\big).\tag{80}$$

By (78), we have

$$\frac{\partial u^i}{\partial x^k}=A^i_k+\frac{\partial u^1}{\partial x^1}\delta^i_k,\tag{81}$$

where the matrix (A^i_k) is skew-symmetric. Now by (80) and (81) we easily get $\widetilde{r}_{ij}=0$. If $n=2$, using (78) we can easily show that $\widetilde{\beta}$ is closed. Then plus $\widetilde{r}_{ij}=0$, $\widetilde{\beta}$ is parallel with respect to $\widetilde{\alpha}$. Q.E.D.

It is shown in [14] that if $n\geq 3$, then all the solutions to (78) are given by

$$u^i=-2\big(\lambda+\langle e,x\rangle\big)x^i+|x|^2e^i+q^i_kx^k+f^i,\tag{82}$$

where λ is a constant number, e,f are constant n-vectors and the constant matrix (q^i_k) is skew-symmetric. For simplicity, let $(q^i_k)=0$ and $e=tf$ for some constant t in (82). It is easy to verify that $\widetilde{\beta}$ determined by (77) and (82) is closed. Then by Lemma 7.1, $\widetilde{\beta}$ is parallel with respect to $\widetilde{\alpha}$. Further, we can verify that if $tf\neq 0$, then $\widetilde{\alpha}$ is of constant sectional curvature if and only if $\lambda^2+t|f|^2=0$. In this case, $\widetilde{\alpha}$ is flat.

Example 7.2 *Defined* α *and* β *by (5), where* $\widetilde{\alpha}$ *and* $\widetilde{\beta}$ *are determined by (77). Let* u *have the following form*

$$u^i = -2\big(\lambda + t\langle f, x\rangle\big)x^i + t|x|^2 f^i + f^i,$$

where t *is a constant and* f *is a constant vector satisfying* $tf \neq 0$ *and* $\lambda^2 + t|f|^2 \neq 0$*. Then the* m*-Kropina metric* $F = \alpha^m \beta^{1-m}$ *is Douglasian but not locally projectively flat, where* $m \neq 0, 1$.

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